Developing Teachers' Knowledge of a Transformations-Based Approach to Geometric Similarity

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U.S. students' poor performance in the domain of geometric transformations is well documented, as are their difficulties applying transformations to similarity tasks. At the same time, a transformations-based approach to similarity underlies the Common Core State Standards for middle and high school geometry. We argue that engaging teachers in this topic represents an urgent but largely unmet need. The article considers what a transformations-based approach to similarity looks like by contrasting it with a traditional, static approach and by providing classroom examples of students using these different methods. In addition, we highlight existing professional development opportunities for teachers in this area.

Key words: Common Core State Standards; Geometry; Professional development; Transformations; Video cases

Geometric Transformations and the Common Core State Standards

Likely among the more challenging recommendations in the recent U.S. Common Core State Standards (CCSS; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) for secondary mathematics teachers are the geometry content standards that encourage a geometric transformations approach to learning congruence and similarity. Beginning in eighth grade, the mathematics standards contain a strong and consistent focus on geometric transformations—including their mathematical properties, how they can be sequenced, and their effect on two-dimensional figures in a coordinate plane. Furthermore, the standards state that congruence and similarity should be defined in terms of rotations, reflections, translations, and dilations.[1]

The current eighth-grade geometry standards include the following:

Understand congruence and similarity using physical models, transparencies, or geometry software.

1. Verify experimentally the properties of rotations, reflections, and translations.
2. Understand that a two-dimensional figure is **congruent** to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
4. Understand that a two-dimensional figure is **similar** to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and *dilations*; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

Supporting both teachers and students in relation to these geometry standards poses a serious challenge for the entire mathematics education community. Much of the content, such as dilation and its relevance to geometric similarity, is likely to be new to teachers. Recently, a national task force of mathematicians, mathematics

1 Dilation, sometimes referred to as a tool for "stretching or shrinking" a figure proportionately, is a geometric transformation that can be applied to objects while maintaining their mathematical similarity.

educators, state leaders, and teacher leaders made recommendations about which CCSS mathematics domains should be targeted as priority areas for professional development and resources in grades K–8. Not surprisingly, they targeted Grade 8 geometry as one of the five recommended priority areas (McCallum, 2011). In their draft report, the task force noted:

> **Geometry in the Common Core State Standards is based on transformations, an approach that is significantly different from previous state standards. This is a change for students, teachers, and teachers of teachers. Challenges include attention to precision and language about transformations. . . . The transformational approach to congruence and similarity is likely unfamiliar to many middle grades teachers. (p. 9)**

For several decades, a small number of mathematicians have posited that all students would greatly benefit from a strong focus on transformations as they learn geometry (Battista, Wheatley, & Talsma, 1982; Clements, 2003; Usiskin, 1972). In their textbook, *Geometry: A Transformational Approach*, Coxford and Usiskin (1971) provide a compelling rationale for the importance of transformations as a central topic in the study of high school geometry. The authors state,

> **Traditional geometry courses have unifying concepts—set, proof—but these are not geometric in nature. The concept of transformation, essential to a mathematical characterization of congruence, symmetry, or similarity, and useful for deducing properties of figures is indeed a unifying concept for geometry.**

Increasingly, mathematicians and mathematics educators have come to agree with this point of view; in fact, they have taken it one step further by advocating for transformations as a unifying topic in the *middle school* geometry curricula (Lappan & Even, 1988; Wu, 2005). As noted, this view of transformations has now been incorporated into the CCSS.

Students' Documented Struggles With Transformations and Similarity

Recent evidence of U.S. middle school students' poor performance in the domain of geometric transformations is provided by the Diagnostic Geometry Assessment project (DGA).[2] DGA has developed an online diagnostic assessment targeting three challenging topics in geometry: shape properties, transformations, and geometric measurement. With respect to transformations, the assessment focuses on student understanding of rotations and reflections with distant centers of rotation and lines of reflection. As part of the DGA project, approximately 900 U.S. middle graders responded to one of two sets of 10 open-ended geometric transformations questions. Their average scores on those assessments were 33% and 37%, depending on the test booklet (Masters, Wing DiMatteo, Nikula, Humez, & Russell, under review). On a new instrument in which these open-ended questions were turned into closed-ended items, an average of only 44% of students answered the items correctly, with the instrument as a whole identifying 59.5% of middle school students as "misconceivers" (Masters, 2010).

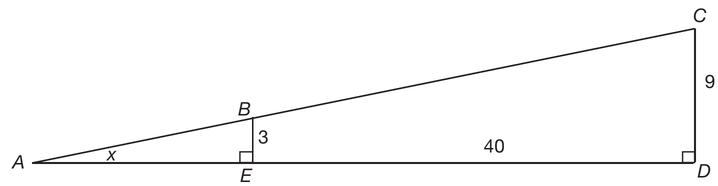
A compelling case that students without a solid understanding of transformations tend to perform poorly on similarity tasks can be made through an examination of relevant NAEP data. For example, the 2007 NAEP item[3] shown in Figure 1 was classified as "Use similarity of right triangles to solve the problem" and was answered correctly by only 1% of eighth-grade students.

Likewise, the 1992 NAEP item shown in Figure 2 was classified as "Find the side length given similar triangles" and was answered correctly by only 24% of high school seniors.

What factors might account for the difficulty an overwhelming majority of 8th graders and even 12th graders have in solving problems involving similar triangles? We conjecture that most students approach these problems using some version of the "traditional," non-transformations-based definition of similarity, such as "corresponding sides are in proportion and corresponding angles are congruent." This type of traditional definition may encourage students to view the NAEP items simply as proportion problems, without attending to the underlying geometric transformations. Even the NAEP rubrics indicate that the problems primarily entail setting up and solving a proportion. The difficulty for students seems to be setting up the proportion, including determining which sides and angles are corresponding and what is being scaled. It is precisely in setting up the proportion where knowledge of geometric transformation is critical. The underlying transformations show clearly which sides and angles correspond and why.

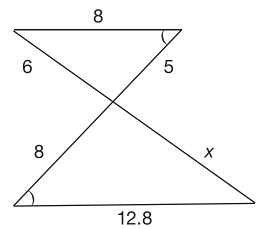
2 More information about DGA is available on the project Web site: https://www.measuredprogress.org/diagnostic-assessments.

3 All the NAEP data discussed in this article can be found on the following website: http://nces.ed.gov/nationsreportcard/itmrlsx/search.aspx?subject=mathematics.



16. The figure above shows two right angles. The length of *AE* is *x* and the length of *DE* is 40.
Show all of the steps that lead to finding the value of *x*. Your last step should give the value of *x*.

Figure 1. **2007 eighth-grade NAEP item.**



9. In the figure above, the two triangles are similar. What is the value of *x*?

Figure 2. **1992 12th-grade NAEP item.**

Thus, a transformations-based approach can support students' conceptual development of proportional and multiplicative reasoning.

For example, in the eighth-grade item shown in Figure 1, comparing the triangles through a dilation leads to guidance for the problem-solver about which sides of the triangles are corresponding and how to scale the sides. Specifically, a dilation with scale factor 3 of the smaller triangle (*ABE*), centered at point *A*, yields the larger triangle in which it is embedded (*ACD*). We know the scale factor is 3 because the side of 3 units is enlarged to a side of 9 units.

The NAEP 12th-grade item shown in Figure 2 can be solved using a rotation combined with a dilation, as demonstrated in Figure 3. The problem states that the triangles are similar, so the challenge for students is to determine which sides of the two triangles are corresponding and determine the scale factor. Here is one way to do this: First, visualize the upper triangle in Figure 3A rotating 180 degrees around the center point in the diagram. The segment marked 6 and the segment marked *x* are collinear, so they will still be collinear after the 180-degree rotation; the same is true for the segment marked 5 and the segment marked 8. The similarity of the two triangles says that they are

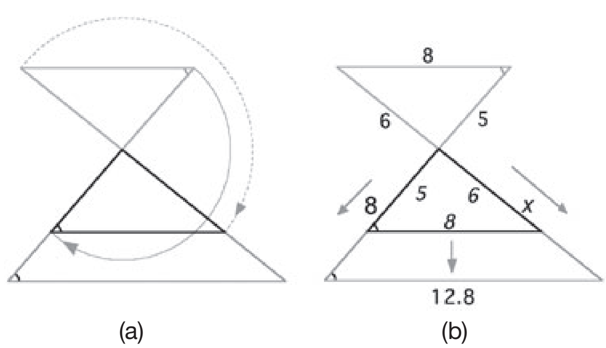


Figure 3. **A geometric transformations approach to the 12th-grade NAEP item shown in Figure 2.**

related by dilation after the rotation. Now, by comparing the corresponding sides in Figure 3B, one can see that the length 5 side of the smaller triangle dilates to a length 8 side of the larger triangle. The length 6 side dilates to a length x side. Therefore, we can generate the proportion $\frac{5}{8} = \frac{6}{x}$.

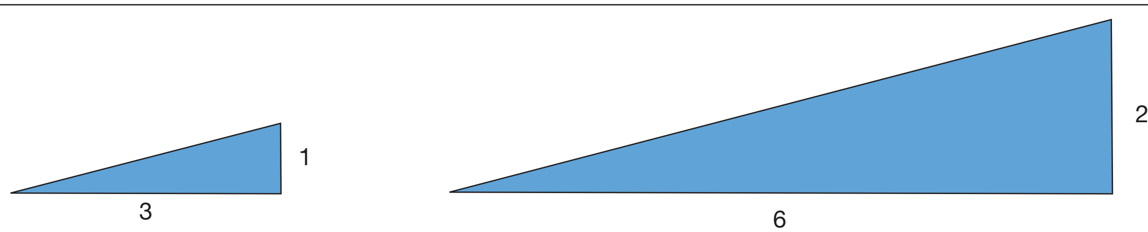
We hypothesize that students who have a strong background in geometric transformations will find relatively little difficulty with items like the ones in Figures 1 and 2. Equipping students with a definition of similarity based on geometric transformations (and congruence as a special case of similarity) is likely to result in a more robust ability to apply this critical mathematical concept. In addition, we conjecture that there are students whose access to mathematics is from a geometric, spatial perspective, and this approach would open the door of opportunity for them mathematically (Ada & Kurtulus, 2010; Clements, Battista, Sarama, & Swaminathan, 1997; Dixon, 1995).

Static and Transformations-Based Perspectives on Similarity

Similarity often is conceptualized in discrete terms as a numeric relationship between two figures, particularly at the secondary school level. We label this a *static perspective*. Figure 4 illustrates the static perspective using similar right triangles. In the figure, the triangles both have side length ratios of 1/3, and the triangle on the right is 2 times the size of the triangle on the left. The triangles can be understood as similar because (looking across triangles) their corresponding side lengths are in the same proportion or because (looking within triangles) the ratio of side lengths within one triangle is equal to the ratio of corresponding side lengths in the other.

By contrast, a *geometric transformations perspective* focuses on enlarging or reducing figures proportionally to create a class of similar figures. An example of a geometric transformations-based definition of similarity is: A figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations. Figure 5 depicts the same 1/3 triangle from Figure 4 and shows how it is related to a class of similar triangles, which can be represented as $1K/3K$. In other words, if the corresponding sides of the triangles are each multiplied by a constant (K), then the ratios of the side lengths are preserved and the triangles are similar. The intention of Figure 5 is to illustrate the notion that these triangles are part of an infinitely large continuous family. Applying a transformations-based definition of similarity, we can say that two figures are similar if scaling one figure produces the other. Alternatively, we can say that a given triangle is congruent to a dilation of all other triangles with the same scale factor, with the center of dilation at the common vertex of those triangles.

Figure 6 shows how the concepts highlighted in Figure 5 can be transferred to a coordinate graph and connected to linearity. Figure 6 illustrates the idea that a class of similar triangles can be represented on a graph and expressed by a specific linear equation. In this case, the linear equation is $y = 1/3x$, where the slope (1/3) is the ratio of the side lengths for the entire class of similar triangles. Figure 6 helps to depict the notion that slope can be used to determine whether figures are similar and represents the continuity of the set of similar figures that can be



$\frac{1}{2} = \frac{3}{6}$ (Corresponding side lengths of similar figures are in proportion) OR

$\frac{1}{3} = \frac{2}{6}$ (Ratios of lengths within a figure are equal to ratios of corresponding lengths in a similar figure)

Figure 4. **Similar triangles from a static perspective.**

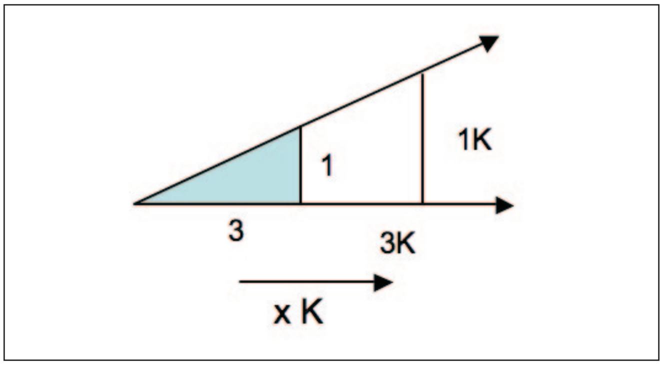


Figure 5. **Similar triangles from a transformations-based perspective. (To see a dynamic version of the figure, click on "similar triangles" to download the file.)**

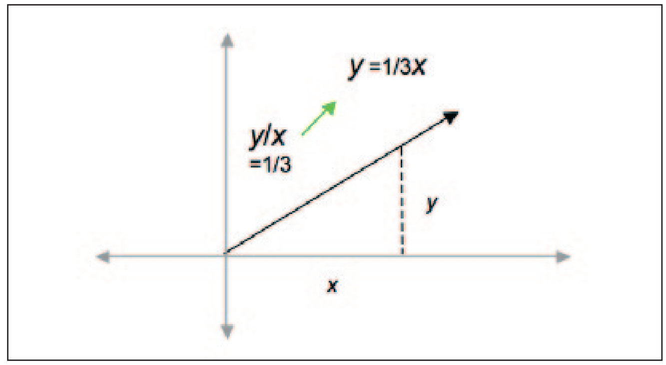


Figure 6. **Connecting similarity and linearity. (To see a dynamic version of the figure, click on "connecting similarity and linearity" to download the file.)**

created through dilation. By making this connection to slope, one gains a sense of the connectedness across branches of mathematics, such as algebra and geometry.

Incorporating Similarity and Transformations Into the Middle School Curriculum

The Learning and Teaching Geometry project. As part of the Learning and Teaching Geometry project (LTG), our research team videotaped several dozen mathematics classrooms (Grades 5–10) across the United States in which teachers implemented similarity problems mostly designed by project staff and occasionally from their own curriculum. A key objective was to obtain images of instruction where transformations played a central role in students' geometric thinking, in order to use those images in professional development materials (Borko, Koellner, Jacobs, & Seago, 2010). In our review of the lessons, we were startled to see the degree to which both teachers and students relied on numerical and measurement strategies to solve these problems, even when they were working in the domain of geometry. Despite purposefully selecting teachers and meeting with them before videotaping to explain our project goals, the majority of their lessons lacked references to transformations-based approaches to problems around similarity. Throughout our work on the LTG project, we found that most of the teachers we encountered did not routinely provide opportunities for their students to make connections between geometric transformations and similarity, and most did not have a solid understanding of a transformations-based approach to similarity themselves.

In the sections that follow, we explore portions of two lessons videotaped for the LTG project in which students are solving problems involving similar rectangles. Our exploration of these segments highlights the distinction between static and transformations-based perspectives, and how students' definitions of similarity come into play as they consider whether given rectangles are similar to one another.

Challenges in solving problems about similar rectangles. Many of the lessons we videotaped for the LTG project involved students working with rectangles, attempting to determine which ones were similar. These classification tasks seemed to contain an appropriate level of challenge and engagement for students in the middle school classrooms that we filmed. Because rectangles are relatively simple shapes, in which all angles measure 90°, all sides are straight, etc., problems involving rectangles provide a straightforward context for students to begin formalizing a definition of similarity.

When working with a set of given rectangles, the question naturally arises: How do you make a determination about which ones are similar if you cannot use angles as evidence?[4] What other criteria should you use? We have found that students can devise many creative ways of correctly and incorrectly answering this question. For example, they might cut the rectangles to see if they fit inside one another, or use the standard "length multiplied by width" algorithm to determine if the rectangles have the same area. This kind of thinking frequently can be generated when students are given rectangles such as 3×4 and 2×6 on grid paper, since each has an area of 12 square units. A critical challenge for teachers is moving students toward a mathematically accurate approach, such as comparing corresponding side lengths, applying a geometric transformation, or both.

4 The angles formed by the sides of a rectangle are always right angles, so they do not help distinguish non-similar rectangles. However, the angle formed by a side and a diagonal of the rectangle do vary for different rectangles and can help distinguish non-similar rectangles.

The Sorting Rectangles Problem: Applying a static perspective. The LTG project team designed the Sorting Rectangles Problem shown in Figure 7 and videotaped several middle school teachers using this problem in their classrooms. In general, students approached the problem from a traditional, static perspective, with relatively limited background knowledge about what similarity means. Here we analyze a portion of a sixth-grade lesson, taught by Pamela, that illustrates this way of thinking.

Pamela's students are seated in groups, and each group is given three bags of four rectangles (see Figure 7). Pamela tells the students that inside each bag is one rectangle that does not belong. She asks them to examine each bag and decide which rectangle doesn't belong and why. She explains that "doesn't belong" means "it's not an enlargement or a reduction."

Two students seated next to each other, Makalya and Victoria, work on Bag B and come up with the same solution. Click here to see video clip 1 (nctm.org/mte/Mak_Vic_Vid1). However, as they talk to one another, it is clear that they attended to different relationships amongst the four rectangles. Their conversation can be paraphrased as follows:

> ***Victoria***
>
> **3 × 2 doesn't belong because notice the other numbers: 2 × 1, 1 is half of 2; 6 × 3, 3 is half of 6; 4 × 2, 2 is half of 4. But 2 is not half of 3.**
>
> ***Makayla***
>
> **3 × 2 doesn't belong because from a 1 × 2 to a 2 × 4 you are adding 1 to the width and 2 to the length. It's the same from a 2 × 4 to a 3 × 6.**

Both Makayla and Victoria are looking at each rectangle in Bag B separately—as discrete figures. Their focus is on comparing the numerical relationships between the corresponding parts of the rectangles. Victoria is noticing that there is a pattern *within* each rectangle (one side is half of the other side) that holds true for three of the four rectangles. Makayla is noticing that there is a pattern *across* the corresponding sides of the rectangles (add 1 to the width and 2 to the length) that holds true for three of the four rectangles. Makayla seems to be approaching the problem in an additive way (e.g., how much is added to the length of the sides), while Victoria seems to be using multiplicative reasoning in noticing a doubling relationship (e.g., how many times bigger is one side compared to another side). Both Makayla and Victoria are correctly applying their understanding of similarity to the Sorting Rectangles problem. Their approaches are valid and accurate; however, they can be contrasted with a transformations-based approach such as the one presented below.

The Rectangle Problem: Applying a transformations-based perspective. The Rectangle problem (see Figure 8) was taught by a different teacher, Hannah, who also was filmed as part of the LTG project. This problem was part

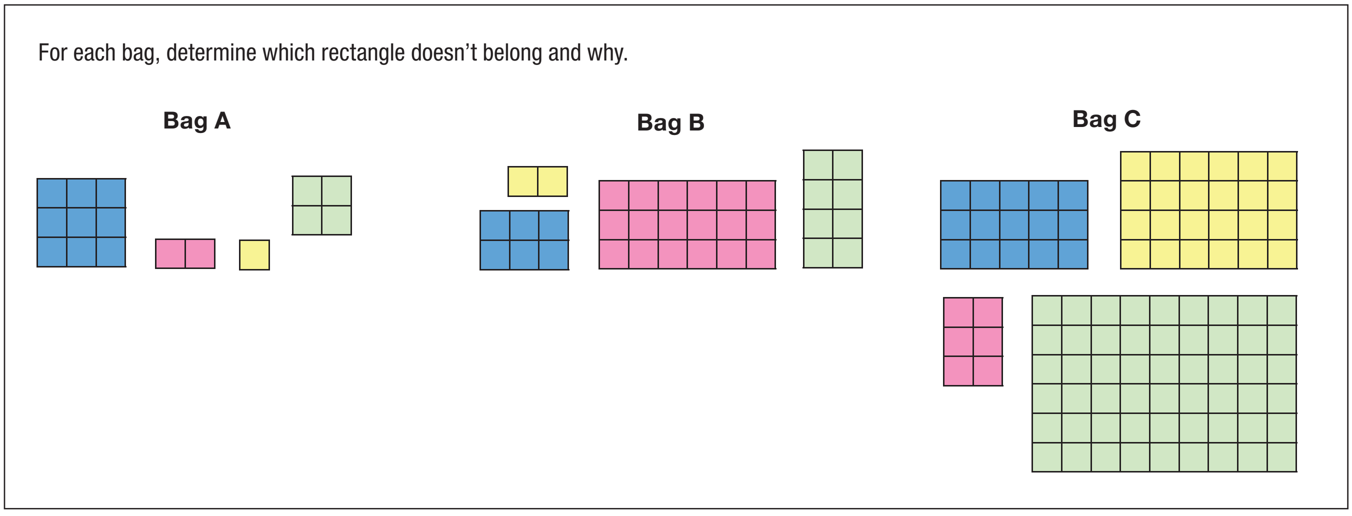


Figure 7. **The Sorting Rectangles Problem.**

Rectangle Problem

Which rectangles are similar to rectangle a? Explain the method you used to decide.

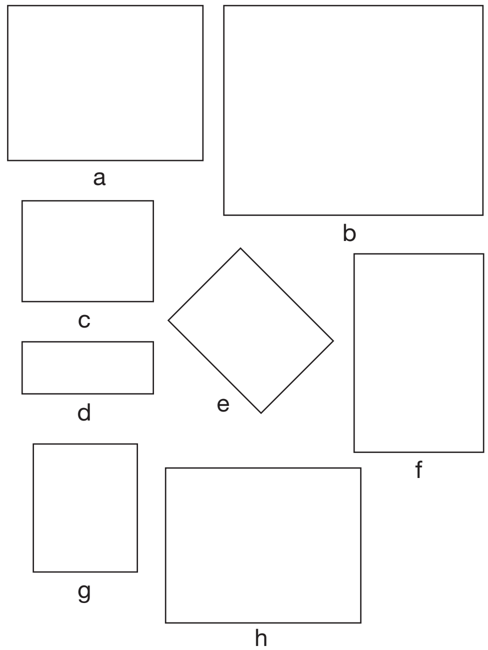


Figure 8. **The Rectangle Problem.**

of Hannah's curricular materials, which were designed to promote a transformations-based approach to similarity.[5] Like the Sorting Rectangles problem, it asks students to examine a set of rectangles and decide which ones are similar. Prior to being videotaped, the students in Hannah's eighth-grade class had a number of experiences working with geometric transformations.

Hannah has her class work on the Rectangle Problem in small groups for about 13 minutes. Then, a few students use the overhead projector at the front of the room to show their strategies to the whole class. Randy shows how he used tracing paper to determine whether the rectangles were dilation images. [Click here to see video clip 2 (nctm.org/mte/Randy_Vid2).](nctm.org/mte/Randy_Vid2) As depicted in Figure 9, first Randy traces rectangle a, and then superimposes it onto rectangle b. He uses a straightedge and draws lines through the corresponding vertices. Randy tells the class that he used the upper left vertex as the "center of dilation," and he noticed that the sides of the two rectangles "lined up" and they shared a common diagonal. Therefore, he correctly concludes, they are similar. Next, applying the same approach, Randy demonstrates that rectangle d is not similar to rectangle a because they do not share a common diagonal.

Randy's approach to the Rectangle Problem draws on an understanding that transformations—in this case dilation—can be used to compare figures and determine whether they are similar. Hannah provided her students with tracing paper, which supported Randy's visual representation of dilating each of the rectangles. We conjecture that Randy used the tracing paper to line up the diagonals of the rectangles in question because he is mentally sliding one rectangle along the diagonal to make the other. In other words, Randy appears to be applying the motion of dilation as a means of proportionally scaling the rectangles. He is selecting one of the corners of the rectangle to be the center of dilation. By the definition of dilation, all points on a line through the center of dilation are sent to other points on this line (and, more specifically, sent to a image whose distance from the center is a fixed scale factor of the distance from the center of dilation to the original point). Hence, all

5 The Rectangle Problem is part of the curriculum materials created by the Curriculum Research & Development Group, University of Hawaii at Mãnoa, http://www.hawaii.edu/crdg/curriculum/.

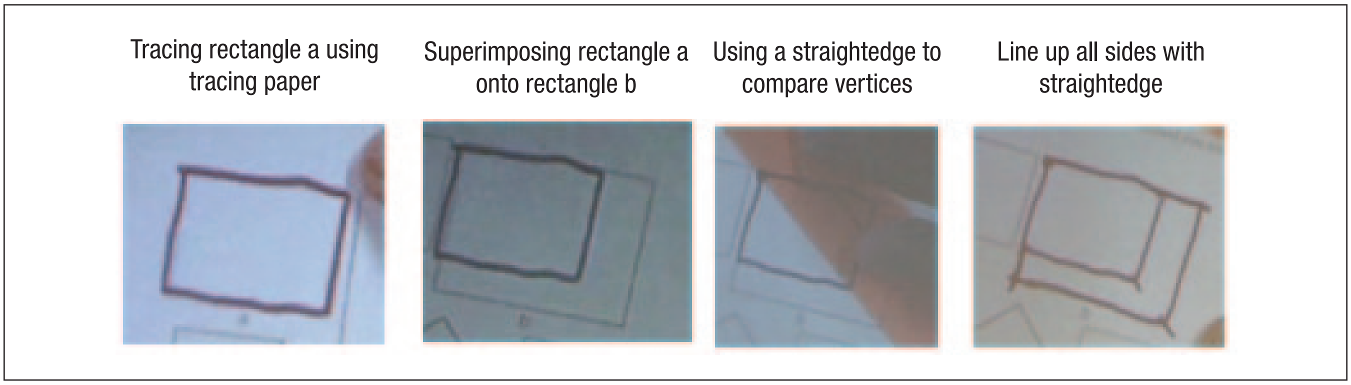


Figure 9. **Randy's dilation method.**

the points on the diagonal of the rectangle are dilated to other points on the line containing this diagonal.

Comparing Victoria's, Makayla's, and Randy's approaches to similarity. Victoria, Makayla, and Randy each take different, but correct, approaches to determining whether given figures are similar. We argue that their approaches are grounded in what can be labeled as either static or transformations-based perspectives. Victoria's and Makayla's approaches are examples of a static perspective; they are attending to the numeric relationships within or across discrete figures. Victoria attends to the relationship within each rectangle, noticing whether the adjacent sides (length and width within the rectangles) are in the same proportion. Makayla attends to the relationships across the corresponding sides, noticing whether they are growing at the same rate (comparing lengths and widths across the figures). Although Victoria and Makayla do not yet attach these types of labels to their own thinking, it is likely that they could come to understand a more formalized expression of their approaches.

Randy's approach is an example of a transformations-based perspective, in which he is attending to the dynamic relationship between the original figure and a family of similar rectangles. Randy constructs dilation lines to determine whether the original figure can be extended in such a way that the corresponding sides and diagonals fall along those dilation lines. Rather than choosing to measure the sides of the rectangles, Randy's approach does not involve any sort of measurement and rather relies on a visual inspection of accurate scaling.

We can examine the three students' approaches to similarity in light of the illustrations presented in Figures 4 and 5, discussed earlier in the article. As indicated in Figure 10, we can place Victoria's and Makayla's approaches within the illustration of a static perspective, and we can place Randy's approach within the illustration of a geometric transformations perspective.

As noted earlier in the article, one extension of a transformations-based approach (exemplified by Randy's method) is to transfer these concepts to a coordinate graph and connect them to linearity (see Figure 6). Specifically, we can build on Randy's approach by creating a set of nested similar rectangles situated on a coordinate graph, and drawing the diagonal through the rectangles from the origin. Figure 11 provides a visual sense of what this connection between similarity, proportional growth, and slope looks like. There is an infinite set of similar rectangles that can be created, constituting a family marked by proportionality, all sharing the same slope.

Promoting Teachers' Understanding of a Transformations-Based Perspective

It is expected that mathematics curriculum, instruction, and assessment across the United States will become increasingly aligned with the CCSS (Achieve, 2010; Phillips & Wong, 2010). In the domain of geometric similarity, this means that teachers will need to understand what a transformations-based perspective looks like and how it contrasts with the more traditional static perspective. We are not arguing that a static approach to similarity is unwarranted or should be replaced by a transformations-based approach. On the contrary, understanding both numeric and geometric relationships among figures is central to mathematics. However, middle graders' prior mathematics experiences are, in many cases, dominated by an emphasis on numerical relationships. We conjecture that by broadening their experiences to incorporate a transformations-based perspective, students will acquire a deeper understanding of concepts related to transformations, congruence, and similarity and will

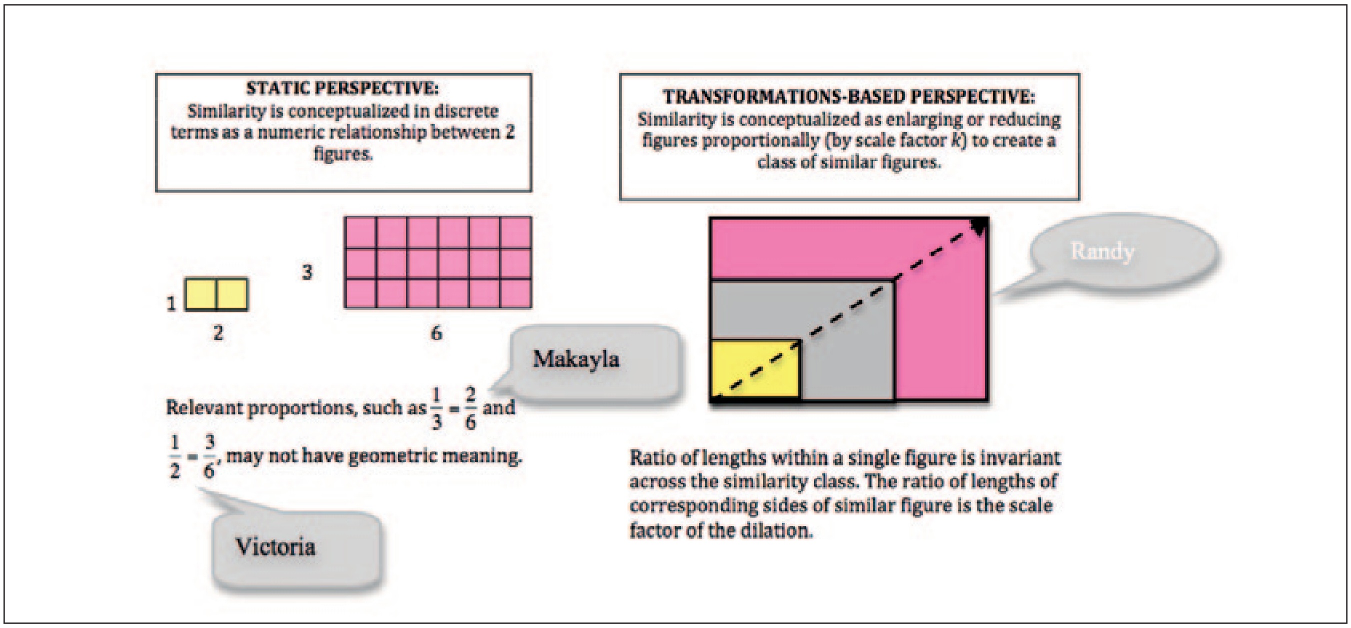


Figure 10. **A comparison of Victoria's, Makayla's, and Randy's approaches.**

be able to more competently apply their knowledge to a wide variety of problem situations. For middle school students, analyzing the effects of geometric transformations on shapes can not only foster insight into mathematical similarity, but may help them develop an intuitive grasp of the meaning of proportionality and bolster their understanding of related concepts, including slope. Students who enter high school with a transformations-based perspective, an understanding of how geometric transformations can determine relationships among geometric figures, and a precise and robust definition of similarity are likely to see important connections across branches of mathematics such as geometry and algebra—and later statistics, trigonometry, and calculus.

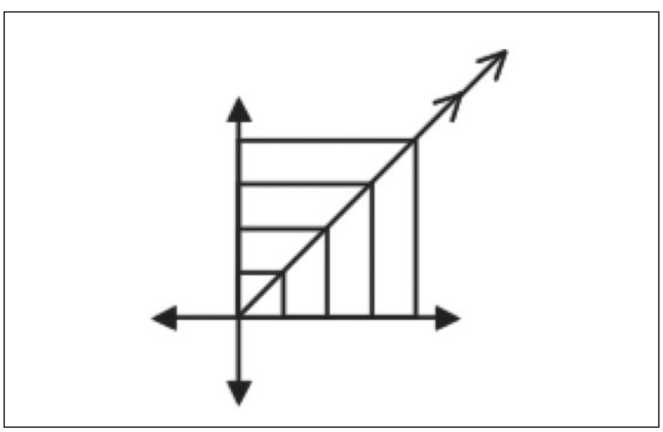


Figure 11. **Nested similar rectangles on a coordinate plane.**

Supporting students in this area will require considerable effort and new knowledge on the part of many mathematics teachers. Teachers will need opportunities to gain "mathematical knowledge for teaching" (Ball & Bass, 2000; Ball & Bass, 2003), including a deep understanding of the mathematical content and the fluency to make instructional decisions that support students' learning of this content. Fostering a geometric transformations perspective means teachers must be able to engage students in geometric thinking and encourage them to apply geometric strategies to solve problems, as opposed to solely arithmetic or measurement strategies. This recommendation is likely to pose a significant challenge for teachers who are faced with teaching new content in ways substantially different from the manner in which they learned it (NCTM, 2006). Although the number of professional development resources designed to help foster teachers' mathematical knowledge for teaching within the topics of number and algebra is increasing (e.g., Barnett, Goldenstein, & Jackson, 1994; Driscoll et al., 2001; Franke, Carpenter, Levi & Fennema, 2001; Schifter et al., 1999a, 1999b; Seago, Mumme, & Branca, 2004; Stein, Smith, & Silver, 2000), far more attention needs to be given to geometry.

The Learning and Teaching Geometry project is devoted to promoting teacher learning of this critical topic. The main goal of the LTG project is to build professional development materials that support the teaching and learning of mathematical similarity from a transformations-based perspective (Seago, Driscoll, & Jacobs, 2010). The LTG project's professional development materials are aligned with the definitions of congruence and similarity outlined in the CCSS; they highlight the role of geometric transformations and the notion that similar figures are part of a continuous family.

The LTG materials engage teachers in learning about similarity through the use of videocases, in which specific and increasingly complex mathematical ideas are presented within the dynamics of classroom practice. Prior to watching a given video clip, teachers grapple with the same mathematical task(s) the videotaped students tackled. They consider issues such as the mathematical concepts entailed in the task and the mathematical reasoning and solution strategies (correct and incorrect) that students are likely to apply to the task. Then, after watching the clip, teachers continue to investigate complex (and often emerging) mathematical ideas. They reflect on how to become more attuned to student thinking and which instructional strategies they might use to foster their students' understanding of the content at hand.

Public access to a portion of the LTG professional development materials. The LTG project has recently collaborated with NCSM and the Noyce Foundation to create two online professional learning modules to support the development of teachers' mathematical knowledge for teaching and promote mathematics instruction that aligns with the Common Core State Standards. The modules, entitled *Congruence and Similarity Through Transformations* and *Similarity, Slope, and Graphs of Linear Functions*, are each designed to be 2-hour teacher learning experiences that illustrate specific standards for mathematical practice. They are presently publicly available on the NCSM website: http://www.mathedleadership.org/ccss/itp/index.html. The modules are designed for use by a knowledgeable facilitator, working face-to-face with a group of teachers. Materials for facilitators include a detailed agenda, PowerPoint slides, time-coded transcripts, lesson graphs, handouts, and a field guide to geometric transformations.

The *Congruence and Similarity Through Transformations* module prompts participants to examine the meaning of defining congruence and similarity through transformations as articulated in the Common Core State Standards. To do this, participants are asked to compare and contrast static definitions of congruence and similarity with transformations-based definitions, as we have discussed in this article. They are also prompted to consider implications for instruction that a transformations-based perspective has on teaching and learning mathematics. During a portion of the module, teachers solve the Rectangle Problem and then watch the videoclip of Randy described earlier. Our experiences piloting the Randy videocase suggest that few teachers initially apply dilation to determine whether the rectangles are the same. In fact, after watching footage of Randy, most teachers need quite a bit of time to process his method. Part of unpacking this videocase with teachers involves asking them to solve the problem again, the same way as Randy. Teachers also consider Randy's mathematical arguments, tool use, and the precision of his language.

We consider the *Congruence and Similarity Through Transformations* module to be an introduction to a transformations-based approach to similarity for teachers. Upon completion of that module, teachers can move on to the *Similarity, Slope, and Graphs of Linear Functions* module, which is based on the premise that a transformations-based approach to similarity provides a useful foundation for exploring slope and graphs of linear functions. During the module, teachers explore the connection between these three mathematical concepts by unpacking a videocase centered on graphing similar rectangles, and then discussing a computer animation that visually displays the main mathematical ideas. Together, these activities serve as the context to give participants an opportunity to gain content knowledge and learn instructional strategies that correspond to a number of practice standards, including supporting students to reason mathematically, to use precise language in their explanations, and to make use of geometric structure.

Carefully examining how students approach similarity tasks is just one example of the types of professional development activities that can foster teachers' understanding of similarity from a transformations-based perspective and support them in effectively applying this knowledge in their classrooms. Certainly the mathematics education community will need to create additional learning experiences for both teachers and students to familiarize them with the nature and importance of a geometric transformations perspective. As the CCSS gain prominence and urgency, developing teachers' knowledge of similarity from a geometric transformations perspective should be on the radar of all mathematics educators.

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